Nonrelativistic anyons in external electromagnetic field

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Abstract

The first-order, infinite-component field equations we proposed before for non-relativistic anyons (identified with particles in the plane with noncommuting coordinates) are generalized to accommodate arbitrary background electromagnetic fields. Consistent coupling of the underlying classical system to arbitrary fields is introduced; at a critical value of the magnetic field, the particle follows a Hall-like law of motion. The corresponding quantized system reveals a hidden nonlocality if the magnetic field is inhomogeneous. In the quantum Landau problem spectral as well as state structure (finite vs. infinite) asymmetry is found. The bound and scattering states, separated by the critical magnetic field phase, behave as further, distinct phases.

1 Introduction

In a previous paper [1] we proposed an infinite set of first-order field equations for non-relativistic anyons, namely

$$\begin{cases} i\partial_t\phi_k + \sqrt{\dfrac{k+1}{2\theta}}\,\dfrac{p_+}{m}\,\phi_{k+1} = 0, \\[2ex] p_-\,\phi_k + \sqrt{\dfrac{2(k+1)}{\theta}}\,\phi_{k+1} = 0, \end{cases} \tag{1.1}$$

k=0,1,.... Here $p_\pm = p_1 \pm ip_2$, and we assume for definiteness that the non-commutative parameter θ (whose physical dimension is m^{-2}, see below) is positive. Grouping the upper and lower equations, respectively, (1.1) can also be presented in the form

$$\begin{cases} D|\phi\rangle = 0 \\ \lambda_-|\phi\rangle = 0 \end{cases}, \qquad \begin{array}{ll} D = i\partial_t - h, & h = \vec{p}\cdot\vec{v} - \tfrac{1}{2}m\,v_+v_-, \\ \lambda_- = p_- - mv_-, & \end{array} \tag{1.2}$$

where $|\phi\rangle = \sum_{k=0}^{\infty}\phi_k|k\rangle_v$. The $|k\rangle_v$ are the Fock states of the velocity operators $v_\pm = v_1 \pm iv_2$, $v_-|0\rangle_v = 0$. The latters are analogous to the α matrices of Dirac, but are associated with an infinite-component, Majorana - type representation of the planar Galilei (rather then Lorentz) group.

The first of these equations is reminiscent of the usual Dirac equation (more precisely, of its non-relativistic counterpart due to Lévy-Leblond [2]) in that it is of the first order in the derivatives. h is the Hamiltonian.

λ_- measures the difference between the canonical (p_i) and the mechanical (mv_i) momenta. The second equation can be viewed as a constraint $\lambda_-|\phi>_{phys}= 0$ which specifies the physical subspace

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as composed of coherent states of the velocity. This makes the spectrum bounded from below by freezing the internal spin degrees of freedom [1].

Eliminating the velocity operator using the constraint converts the first equation into a quadratic expression, which is the quantum version of the acceleration-dependent system of Lukierski et al. [3], and can be obtained as a tricky non-relativistic limit of Polyakov's "particle with torsion" [4, 5].

The system (1.2) realizes the "exotic" [i. e., two-fold centrally extended] planar Galilean symmetry [6]. The commutator of the velocity operators is in fact $[v_-, v_+] = 2\kappa^{-1}$ where the real constant $\kappa = \theta m^2$ is indeed the second central charge that measures the extent Galilean boosts fail to commute [6]. The Hamiltonian and the constraint weakly commute,

$$\Big[h, \lambda_-\Big] = \frac{1}{m\theta}\lambda_-. \tag{1.3}$$

The relation (1.3), which is the archetype of "good" behaviour, guarantees that the physical states do not mix with the unphysical ones during time evolution, i.e., the *consistency* of the system.

The equations (1.2) only describe free particles, though, and coupling them to electromagnetic field is not entirely trivial. The standard "minimal coupling" rule $\vec{p} \to \vec{p} - e\vec{A}$, simply inserted into both the Hamiltonian and the constraint, yields in fact also some unwanted terms, see (2.10) below.

We would also like to remind the reader to the analogous difficulties encountered in the relativistic case: the first-order, Majorana-Dirac type anyon field equations considered in [7, 8, 9] as well as those put forward by Dirac [10], only describe free particles. Their coupling to external electromagnetic field is still an unsolved problem.

In this paper, we find a Hamiltonian and a constraint which involve the electromagnetic field and such that they *weakly commute*.

A general framework, which includes both minimal and also nonminimal coupling, is presented. Special attention is paid to a critical case. The quantum Landau problem and the associated field equations are studied in detail.

Our paper is organized as follows. In Section 2 the free classical symplectic model which underlies our field-theoretical system (1.2) is generalized so that it accommodates arbitrary background magnetic and electric fields. The system we obtain is described by two second class constraints, which, at the critical value of magnetic field, are transmuted into first class constraints.

Section 3 is devoted to the analysis of classical system in the critical case. It is performed at the level of the equations of motion, proceeding from the generic case and developing a Hamiltonian analysis. For minimal coupling the reduced Hamiltonian is given by the initial scalar potential corrected by a term, which is quadratic in the electric field.

In Section 4 our analysis is extended to the quantum case. First we identify the quantum Hamiltonian and constraints, and then show that, when the magnetic field is inhomogeneous, the theory reveals a hidden nonlocal structure.

Then we analyse in detail the Landau problem in a constant B- field, for which theory is local, and the generalization of the free anyon field equations (1.1) are found.

A special section is devoted to the critical case, and its relation to the generic (noncritical) Landau problem.

Our results are summarized in Section 6.

2 Classical framework

2.1 The free symplectic model

The classical counterpart of the system (1.2) is given by its symplectic structure and a pair of real second class constraints

$$\omega = dp_i \wedge dx_i + \frac{1}{2}\kappa\epsilon_{ij} dv_i \wedge dv_j, \tag{2.1}$$

$$\lambda_i = p_i - mv_i \approx 0, \qquad \{\lambda_i, \lambda_j\} = -\theta^{-1}\epsilon_{ij}, \tag{2.2}$$

augmented with a Hamiltonian, presented in either of the equivalent forms

$$h = p_i v_i - \frac{1}{2} m v_i^2 = \frac{1}{2} m v_i^2 + v_i \lambda_i = \frac{1}{2m}(p_i^2 - \lambda_i^2). \tag{2.3}$$

The Lagrange multipliers (v_i) in the middle expression guarantee the conservation of the constraints, $\dot{\lambda}_i = \{\lambda_i, h\} \approx 0$ [1]. The system has the conserved angular momentum

$$J = \epsilon_{ij} x_i p_j + \frac{1}{2} \theta m^2 v_+ v_-. \tag{2.4}$$

The second class constraints (2.2) reduce the number of physical phase space degrees of freedom from 6 to 4. The gauge-invariant extension of the original coordinates x_i,

$$X_i = x_i - \theta \epsilon_{ij} \lambda_j, \tag{2.5}$$

and the momenta satisfy the relations $\{X_i, \lambda_j\} = \{p_i, \lambda_j\} = 0$, and can be therefore identified with dynamical variables describing the physical degrees of freedom (observables) of the system. By (2.1),

$$\{p_i, p_j\} = 0, \quad \{X_i, p_j\} = \delta_{ij}, \quad \{X_i, X_j\} = \theta \epsilon_{ij}. \tag{2.6}$$

Let us stress, in particular, that the new coordinates X_i are non-commuting. The angular momentum is presented equivalently as

$$J \approx \epsilon_{ij} X_i p_j + \frac{1}{2} \theta p_i^2. \tag{2.7}$$

The Poisson bracket relations (2.6) imply that, when restricted to the surface of second class constraints, the classical system is equivalent to the free exotic particle of [11].

2.2 Interactions

Let us assume that the magnetic and electric fields are static, given by abelian vector and scalar potentials, $A_i(\vec{x})$ and $V(\vec{x})$, $B = \epsilon_{ij} \partial_i A_j$, and $eE_i = -\partial_i V$, respectively. Let us consider the usual minimal coupling rule

$$p_i \to P_i = p_i - eA_i(\vec{x}). \tag{2.8}$$

Inserting (2.8) into the Hamiltonian and the constraints,

$$h \to \widetilde{H} = \vec{P} \cdot \vec{v} - \frac{1}{2} m \vec{v}^{\,2} + V(\vec{x}) \qquad \text{and} \qquad \lambda_i \to \widetilde{\Lambda}_i = P_i - m v_i \approx 0, \tag{2.9}$$

respectively, would yield additional terms in the Poisson bracket,

$$\left\{\widetilde{H}, \widetilde{\Lambda}_i\right\} = \frac{1}{m\theta} \widetilde{\Lambda}_i + eBv_i + \partial_i V. \tag{2.10}$$

violating the consistency relation (1.3).

Similarly, $\{P_i, \widetilde{\Lambda}_j\} = eB\epsilon_{ij} \neq 0$ so that the P_i in (2.8) are not observable.

Below we correct these defects. On account of

$$\{\widetilde{\Lambda}_i, \widetilde{\Lambda}_j\} = -\theta^{-1}(1-\beta)\epsilon_{ij} \qquad \text{where} \qquad \beta = \beta(\vec{x}) = e\theta B(\vec{x}), \tag{2.11}$$

it is necessary to distinguish two cases. Let us define the critical magnetic field by putting

$$B_c = \frac{1}{e\theta}. \tag{2.12}$$

When $B \neq B_c$, the constraints (2.9) are second class, but for $B = B_c$ they turn into first class. These cases should be analyzed separately.

[1] The notation $\approx$ means "on-shell" i. e. after restriction to the constrained surface.

We first consider the generic case $B \neq B_c$. Let us first assume that we only have a magnetic field, and try to generalize the free (kinetic) Hamiltonian in (2.3) as

$$H_B = \frac{1}{2} m v_i^2 + u_i \widetilde{\Lambda}_i. \tag{2.13}$$

Requiring that the constraints (2.9) be preserved, $\frac{d}{dt}\widetilde{\Lambda}_i \approx 0$, fixes the Lagrange multipliers as $u_i = (1-\beta)^{-1} v_i$, and we get

$$H_B = \frac{1}{2} m v_i^2 + v_i \Lambda_i, \qquad \text{where} \qquad \Lambda_i = \frac{1}{1-\beta}\,\widetilde{\Lambda}_i \approx 0. \tag{2.14}$$

Then

$$\{H_B, \Lambda_i\} = \left(\left(\frac{1}{m\theta}\delta_{jl} + \frac{1}{2} v_j \epsilon_{lk} \partial_k\right)(1-\beta(x))^{-1}\right)\Lambda_l \epsilon_{ji}$$

weakly vanishes, as expected.

As long as $\beta \neq 1$, the new constraints Λ_i in (2.14) are equivalent to the old ones in (2.9) and indeed satisfy

$$\{\Lambda_i, \Lambda_j\} = \left(-\frac{1}{\theta(1-\beta)} + \Delta\right)\epsilon_{ij} \approx -\frac{1}{\theta(1-\beta)}\,\epsilon_{ij}, \qquad \Delta = \frac{1}{2}\epsilon_{ij}\Lambda_i \partial_j (1-\beta)^{-1} \approx 0. \tag{2.15}$$

The Λ_i can also be presented in a form similar to (2.2),

$$\Lambda_i = \mathcal{P}_i - m v_i \approx 0 \qquad \text{with} \qquad \mathcal{P}_i = \frac{1}{1-\beta}\,(P_i - m\beta v_i). \tag{2.16}$$

The new "momenta" $\mathcal{P}_i$, unlike the P_i, *are* observable, $\{\mathcal{P}_i, \Lambda_j\} = \epsilon_{ij}\Delta \approx 0$. The Hamiltonian (2.14) has again equivalent forms, namely

$$H_B = \mathcal{P}_i v_i - \frac{1}{2} m v_i^2 = \frac{1}{2m}(\mathcal{P}_i^2 - \Lambda_i^2). \tag{2.17}$$

The generalization of the free coordinate X_i in (2.5),

$$\mathcal{X}_i = x_i - \theta \epsilon_{ij} \Lambda_j, \tag{2.18}$$

is also observable, $\{\mathcal{X}_i, \Lambda_j\} = \theta\Delta\delta_{ij} \approx 0$. Putting $\Omega = (1-\beta)^{-1} + \theta\Delta$,

$$\left\{\mathcal{X}_i, \mathcal{X}_j\right\} = \theta\Omega\,\epsilon_{ij} \approx \frac{\theta}{1-\beta}\,\epsilon_{ij}, \tag{2.19}$$

$$\left\{\mathcal{X}_i, \mathcal{P}_i\right\} = \Omega\,\delta_{ij} \approx \frac{1}{1-\beta}\,\delta_{ij}, \tag{2.20}$$

$$\left\{\mathcal{P}_i, \mathcal{P}_j\right\} = \frac{\Omega - 1}{\theta}\,\epsilon_{ij} \approx \frac{eB}{1-\beta}\epsilon_{ij}. \tag{2.21}$$

The angular momentum (2.7) is now generalized to

$$J = \epsilon_{ij}\mathcal{X}_i \mathcal{P}_j + \frac{1}{2}\theta\mathcal{P}_i^2 + \frac{1}{2} eB\mathcal{X}_i^2. \tag{2.22}$$

It generates rotations of the observables $\mathcal{X}_i$ and $\mathcal{P}_i$.

Having identified the observable variables which correspond to the physical degrees of freedom, now we extend the Hamiltonian in (2.14) by adding a scalar potential.

- Let us first consider

$$H = H_B + V(\mathcal{X}), \tag{2.23}$$

for which the (1.3)-type consistency condition $\left\{H, \Lambda_i\right\} \approx 0$ holds.

Since $\mathcal{X}_i \approx x_i$, the derivative $\partial_i V(\mathcal{X}) = -eE_i(\mathcal{X})$ is ($-e$ times) the electric field. (2.23) is viewed therefore as a generalization of the minimally coupled Hamiltonian. It generates the equations of motion

$$m^*\dot{\mathcal{X}}_i = \big(\big(\mathcal{P}_i + m\theta\epsilon_{ij}\partial_j V(\mathcal{X})\big)\,\Omega - \theta\Delta\Lambda_i\big)\,(1-\beta) \approx \mathcal{P}_i - em\theta\epsilon_{ij}E_j, \tag{2.24}$$

$$m^*\dot{\mathcal{P}}_i = \theta^{-1}\epsilon_{ij}\big(m^*\dot{\mathcal{X}}_j - (1-\beta)\mathcal{P}_j\big) \approx eB\epsilon_{ij}\mathcal{P}_j + emE_i. \tag{2.25}$$

where

$$m^* = m(1-\beta) = m(1-e\theta B) \tag{2.26}$$

is the effective mass. The second equation from (2.25) on account of the first one can be presented in the equivalent form

$$\dot{\mathcal{P}}_i \approx eB(\mathcal{X})\epsilon_{ij}\dot{\mathcal{X}}_j + eE_i(\mathcal{X}), \tag{2.27}$$

On-shell, our extended scheme reduces hence to that of [11].

Off the critical case, the variables $\mathcal{X}_i, \mathcal{P}_i$ are observable and provide us with a satisfactory description of the system in terms of the constraints Λ_i (2.16) and the Hamiltonian H (2.23). The interaction can, however, also be discussed in terms of the vector

$$Y_i = x_i + m\theta\epsilon_{ij}v_j, \tag{2.28}$$

whose use will be particularly convenient in the critical case. In terms of $\mathcal{X}_i$ and $\mathcal{P}_i$, it can be presented equivalently as

$$Y_i = \mathcal{X}_i + \theta\epsilon_{ij}\mathcal{P}_j. \tag{2.29}$$

Y_i is, like $\mathcal{X}_i$ and $\mathcal{P}_i$, observable, but unlike these, it *strongly* commutes with the constraints,

$$\{Y_i, \Lambda_j\} = 0. \tag{2.30}$$

Y_i is decoupled from the coordinates $\mathcal{X}_i$ and satisfies

$$\{Y_i, \mathcal{X}_j\} = 0, \qquad \{Y_i, \mathcal{P}_j\} = \delta_{ij}, \tag{2.31}$$

$$\{Y_i, Y_j\} = -\theta\epsilon_{ij}. \tag{2.32}$$

In terms of $\mathcal{P}_i$ and Y_i, (2.24) takes the form familiar from point mechanics,

$$m\dot{Y}_i \approx \mathcal{P}_i. \tag{2.33}$$

Using the decoupled, independent observables $\mathcal{X}_i$ and Y_i, the generator of rotations, (2.22), can be represented in a quadratic normal form,

$$J = \frac{1}{2\theta}\left(Y_i^2 - (1-\beta)\mathcal{X}_i^2\right). \tag{2.34}$$

- Another Hamiltonian can also be considered now:

$$\check{H} = H_B + \check{V}(Y). \tag{2.35}$$

where, to avoid confusion with the previous case, we called the potential $\check{V}$.

It also satisfies the classical counterpart of (1.3), $\{\check{H}, \Lambda_i\} \approx 0$. The expansion

$$\check{V}(Y) = \check{V}(\mathcal{X}) - \theta e\epsilon_{ij}\check{E}_i(\mathcal{X})\mathcal{P}_j + \dots$$

allows us to infer that $\partial_i\check{V}(Y) = -e\check{E}_i(Y)$ is the electric field only if $\check{V}(Y)$ is linear. The Hamiltonian (2.35) describes therefore a particle with non-commuting coordinates and with non-minimal coupling. The associated equation of motion can be written in the form

$$m^*\dot{\mathcal{X}}_i \approx \mathcal{P}_i, \qquad m^*\dot{\mathcal{P}}_i \approx eB\epsilon_{ij}\mathcal{P}_j + em^*\check{E}_i(Y), \tag{2.36}$$

cf. Eqs. (2.33) and (2.25), respectively[2].

For a constant magnetic field $B = B_0 \neq B_c$, the Lorentz force law (2.27) of the minimal coupling case can be presented in the equivalent form

$$\dot{\mathcal{X}}_i^{gc} \approx \frac{1}{B_0}\epsilon_{ij}E_j(\mathcal{X}), \tag{2.37}$$

where

$$\mathcal{X}_i^{gc} := \mathcal{X}_i + \frac{1}{eB_0}\epsilon_{ij}\mathcal{P}_j. \tag{2.38}$$

In the nonminimal case we have instead, using (2.36),

$$\dot{\mathcal{X}}_i^{gc} \approx \frac{1}{B_0}\epsilon_{ij}\check{E}_j(Y). \tag{2.39}$$

$\mathcal{X}_i^{gc}$ can therefore be interpreted, in both cases, as the guiding center coordinate [13].

Compared to Y_i, $\mathcal{X}_i^{gc}$ behaves in the opposite way: it is decoupled from $\mathcal{P}_i$ (but not from $\mathcal{X}_i$), $\{\mathcal{X}_i^{gc}, \mathcal{P}_j\} \approx 0$ [that is behind its evolution law (2.37) or (2.39)], and its brackets depend on the value of the constant magnetic field,

$$\{\mathcal{X}_i^{gc}, \mathcal{X}_j^{gc}\} \approx -\frac{1}{eB_0}\epsilon_{ij}. \tag{2.40}$$

Let us stress, however, that the definition (2.38) of the guiding center coordinate is restricted to a homogeneous magnetic field. Naively extended to the inhomogeneous case, $\mathcal{X}_i^{gc}$ will perform circular motion with amplitude proportional to the gradient of B, i.e. it will be a guiding center coordinate in zero order of $\partial_i B$. The brackets (2.40) also will be corrected by the term proportional to the gradient of magnetic field.

We conclude this section with presenting the second-order form of our equations of motion.

- In the minimal case (2.24) – (2.25) we have

$$\frac{d}{dt}\big(m^*\dot{\mathcal{X}}_i + m\theta e\epsilon_{ij}E_j(\mathcal{X})\big) \approx eB(\mathcal{X})\epsilon_{ij}\dot{\mathcal{X}}_j + eE_i(\mathcal{X}). \tag{2.41}$$

Expressed in terms of the coordinates Y_i, this reads

$$m^*\ddot{Y}_i \approx eB(\mathcal{X})\epsilon_{ij}\dot{Y}_j + eE_i(\mathcal{X}), \tag{2.42}$$

where the arguments of the magnetic and electric fields are given by $\mathcal{X}_i = Y_i - m\theta\epsilon_{ij}\dot{Y}_j$.

- In the non-minimal case (2.35) we have instead

$$\frac{d}{dt}(m^*\dot{\mathcal{X}}_i) \approx eB(\mathcal{X})\epsilon_{ij}\dot{\mathcal{X}}_j + e\check{E}_i(Y), \tag{2.43}$$

where it is assumed that $Y_i = \mathcal{X}_i + m^*\theta\epsilon_{ij}\dot{\mathcal{X}}_j$. Equivalently, in terms of the Y_i,

$$m^*\frac{d}{dt}(\dot{Y}_i - e\theta\epsilon_{ij}\check{E}_j(Y)) \approx eB(\mathcal{X})\epsilon_{ij}\dot{Y}_j + e\check{E}_j(Y), \tag{2.44}$$

where it is assumed that the argument of B is expressed via the relation $\mathcal{X}_i = Y_i - m\theta\epsilon_{ij}\dot{Y}_j - m\theta^2\check{E}_i(Y)$.

Note that for constant magnetic and electric fields, $B = B_0 = const$, and $E_i = \check{E}_i = E_i^0 = const$, respectively, the equations (2.41), (2.42), (2.43) and (2.44) take all the same form, namely

$$m^*\ddot{Z}_i = eB_0\epsilon_{ij}\dot{Z}_j + eE_i^0, \tag{2.45}$$

where Z_i is either $\mathcal{X}_i$ or Y_i.

[2]The similarity between (2.36) and (2.33) suggests a kind of "duality" between the two types of couplings. Another generalization includes anomalous coupling [12].

3 The critical case $B = B_c$

For $B = B_c$ the constraints Λ_i as well as the observables $\mathcal{P}_i$ and $\mathcal{X}_i$ are all ill-defined; the variable Y_i in (2.28) behaves in turn regularly.

The guiding center coordinates (2.38) are, a priori, only defined off the critical case. The divergences are readily seen to cancel as $B_0 \to B_c$, however, and (2.38) becomes precisely Y_i, presented in the form (2.29). The brackets (2.40) take the form (2.32). Thus, in the critical case, Y_i becomes the guiding center coordinate.

• For the minimal coupling, (2.23) at $B = B_c$, Eq. (2.24) is reduced to $\mathcal{P}_i \approx em\theta\epsilon_{ij}E_j$, and Y_i becomes, using Eq. (2.29) and $\mathcal{X}_i \approx x_i$,

$$Y_i \approx x_i - \frac{m}{eB_c^2}E_i(\vec{x}). \tag{3.1}$$

The equations of motion (2.42) become now first order,

$$\dot{Y}_i \approx \frac{1}{B_c}\epsilon_{ij}E_j(\vec{x}), \tag{3.2}$$

cf. Eqn. (2.37), and we indeed recognize Y_i as the familiar expression of the guiding center in the Hall effect.

• Similarly in the non-minimal case (2.35), at $B = B_c$ the first equation from (2.36) gives $\mathcal{P}_i \approx 0$, and with $\mathcal{X}_i \approx x_i$, from Eqn. (2.29) we find

$$Y_i \approx x_i. \tag{3.3}$$

Then, on account of Eq. (2.43) with $m^* = 0$, we find that the *guiding center* follows the law

$$\dot{Y}_i \approx \frac{1}{B_c}\epsilon_{ij}\check{E}_j(Y), \tag{3.4}$$

cf. Eqn. (2.39).

Both equations (3.2) and (3.4) are reminiscent of the *Hall law*, $\dot{Z}_i = \frac{1}{B_c}\epsilon_{ij}E_j^0$, to which they both reduce if the electric field is homogeneous. In the general case, however, they are slightly different : on the one hand, in (3.2) the argument of electric field is x_i, which is related to Y_i via Eqn. (3.1). On the other hand, as noted above, $\check{E}_i$ in (3.4) is the electric field only for $\check{E}_i = const.$

After these preliminary observations, we present a Hamiltonian analysis of the critical case.

For $B = B_c$ we have $\beta = 1$; the constraints (2.9) become first class, and reduce therefore the dimension of the physical phase subspace by 4, rather than by 2. In all cases (critical or not), $\{Y_i, \tilde{\Lambda}_j\} = 0$. In the critical case the noncommuting variables Y_i represent the two independent phase space degrees of freedom of the physical subspace. As already said, the relation of the initial coordinates x_i to Y_i depends on the choice of the potential.

For the consistency of the theory, the conservation of the constraints (2.9) has to be checked.

• Let us first consider the minimally coupled system with $B = B_c$. We seek again our Hamiltonian in the form

$$H = \frac{1}{2}mv_i^2 + u_i\tilde{\Lambda}_i + V(\vec{x}), \tag{3.5}$$

The first two terms here are as in (2.13), but the argument of the potential has been changed from $\mathcal{X}_i$ [which is ill-defined for $B = B_c$] to $x_i (\approx \mathcal{X}_i)$. The conservation of the constraints $\tilde{\Lambda}_i \approx 0$ results in the gauge-fixing conditions

$$\chi_i = v_i + \theta\epsilon_{ij}\partial_j V(\vec{x}) \approx 0, \qquad \text{i. e.} \qquad P_i \approx m\epsilon_{ij}\frac{E_j(\vec{x})}{B_c} \tag{3.6}$$

which played a rôle in the Hamiltonian reduction in [11].

The conservation of (3.6) requires in turn,

$$M_{ij}u_j = v_i, \qquad M_{ij} = \delta_{ij} + m\theta^2\partial_i\partial_j V. \tag{3.7}$$

When the matrix M_{ij} is non-singular, the constraints (2.9) and gauge conditions (3.6) provide us with four second class constraints; then the equations (3.7) can be solved for the u_i.

Let us mention for completeness that, for a repulsive oscillator potential

$$V(x) = -\alpha\frac{eB}{2m\theta}x_i^2 + \mu_i x_i + \nu,$$

where α, μ_i and ν are constants, the matrix M_{ij} in (3.7) vanishes if $\alpha = 1$. Then (3.7) gives new constraints $v_i \approx 0$, and we get 6 second class constraints and reduction yields a zero-dimensional phase subspace (i. e. a point) with fixed values

$$x_i = Y_i = m\theta^2\mu_i, \quad p_i = eA_i^c(x)_{x_i=m\theta^2\mu_i}, \quad \epsilon_{ij}\partial_i A_j^c = B_c.$$

This can be understood as follows. The equations of motion (2.41) with $V(\mathcal{X}) = V(x)_{x_i=\mathcal{X}_i}$ specified above, now read

$$(1-\beta)\ddot{\mathcal{X}}_i + (\alpha-1)\frac{eB}{m}\epsilon_{ij}\dot{\mathcal{X}}_j - \frac{\alpha}{m\theta}\frac{eB}{m}\mathcal{X}_i + \frac{1}{m}\mu_i = 0. \tag{3.8}$$

When $B = B_c = (e\theta)^{-1}$ and $\alpha \neq 1$, we have the Hall-like law (3.2) [whereas for $\alpha = 1$, eq. (3.8) reduces to $\mathcal{X}_i = m\theta^2\mu_i$]. When $B \neq B_{crit}$ and $\alpha = 1$, for $\beta(1-\beta) < 0$ and $\beta(1-\beta) > 0$, the system performs rotational resp. hyperbolic motion around the point $\mathcal{X}_i = m\theta^2\mu_i$. Hence, $\alpha = 1$, $B = B_c$ corresponds to the boundary that separates these two phases.

Returning to the generic case, (3.6) says that the variables v_i are determined by the Hall law,

$$v_i \approx \epsilon_{ij}\frac{E_i(\vec{x})}{B_c}. \tag{3.9}$$

The interpretation of this relation requires some care, however : in the coupled case we consider here, the variables v_i do *not* represent anymore the time derivative of the original position. Assuming that $M = (M_{ij})$ is non-singular, the "velocity" equation reads in fact

$$\dot{x}_i = u_i = \left(M^{-1}\right)_{ij} v_j. \tag{3.10}$$

On the other hand, $\dot{Y}_i = \{H, Y_i\} = v_i$, which identifies v_i as the time derivative of the guiding center coordinate Y_i; this latter satisfies the Hall-like law, (3.2).

On account of the relation $(M^{-1})_{ij} = (\det M)^{-1}\epsilon_{ik}\epsilon_{jl}M_{kl}$, equations (3.10) with v_i given by (3.9) can be presented in Hamiltonian form,

$$\dot{x}_i = \{x_i, H_c\} \qquad \text{with} \quad H_c = V(\vec{x}) + \frac{m}{2}\theta^2(\partial_i V(\vec{x}))^2 \tag{3.11}$$

and

$$\{x_i, x_j\} = -\theta(\det M)^{-1}\epsilon_{ij}, \qquad \det M = 1 + m\theta^2\partial_i^2 V + \frac{1}{2}(m\theta^2)^2\epsilon_{kl}\epsilon_{rs}(\partial_k\partial_r V)(\partial_l\partial_s V). \tag{3.12}$$

The Hamiltonian (3.11) is indeed the reduction of (3.5) to the surface defined by the second class constraints (2.9) and (3.6), with (3.12) the corresponding Dirac brackets. Note that in the Landau problem ($V = 0$), Eq. (3.1) is reduced to $Y_i \approx x_i$, and the brackets (3.12) coincide with those of Y_i.

Alternatively, these coordinates Y_i can be used to describe also the system reduced to the surface (2.9), (3.6). Their brackets are $\{Y_i, Y_j\} = -\theta\epsilon_{ij}$ cf. (2.31), and the dynamics (3.2) is generated by the Hamiltonian

$$H_c = \mathcal{V}(Y) \qquad \text{with} \qquad \mathcal{V}(Y) = \left(V(\vec{x}) + \frac{m}{2}\theta^2(\partial_i V(\vec{x}))^2\right)_{x_i = x_i(Y)}, \tag{3.13}$$

where $x_i(Y)$ is given by (3.1). This also explains the advantage of quantizing the system in terms of the Y_i. The reduced Hamiltonian H_c extends the rule called " Peierls substitution" [14] to the non-commutative case. The $V(Y)$ alone used in [11], obtained dropping the θ-term, is only correct for constant fields.

- The non-minimal Hamiltonian is instead

$$\check{H} = \frac{1}{2}mv_i^2 + u_i\tilde{\Lambda}_i + \check{V}(Y). \tag{3.14}$$

The conservation of the first class constraints requires now

$$\chi_i = v_i \approx 0. \tag{3.15}$$

The functions χ_i are such that $\det ||\{\varphi_a, \varphi_b\}|| \neq 0$, where φ_a, $a = 1, 2, 3, 4$, are $\varphi_a = (\tilde{\Lambda}_i, \chi_j)$. (3.15) is a gauge-fixing for the constraints (2.9) [15]; the conservation of (3.15) fixes furthermore the Lagrange multipliers as

$$u_i = v_i + \theta\epsilon_{ij}\partial_j\check{V}(Y) \approx \theta\epsilon_{ij}\partial_j\check{V}(Y).$$

Reduced to the surface given by the set of second class constraints (2.9) and gauges (3.15), the system is described by the Hamiltonian

$$\check{H}_c = \check{V}(Y). \tag{3.16}$$

It follows from $\{Y_i, \tilde{\Lambda}_j\} = 0$, that the Dirac bracket of the reduced phase space coordinates $Y_i = x_i$ coincides with their initial Poisson bracket (2.32). The equation of motion of the system (3.14) is therefore (3.4). With hindsight to Eq. (2.34), note that in the critical case the generator of rotations is reduced to

$$J = \frac{1}{2\theta}Y_i^2. \tag{3.17}$$

4 Quantization and field equations

Now we quantize the coupled system. The different behaviour of the constraints $\widetilde{\Lambda}_i \approx 0$ and $\Lambda_i \approx 0$ for $B \neq B_c$ and $B = B_c$, respectively, obliges us to distinguish between these two cases also at the quantum level. As we observed in Section 2.2, the $\widetilde{\Lambda}_i$ are classically equivalent to Λ_i for $B \neq B_c$, but, unlike the latters, the $\widetilde{\Lambda}_i$ are well defined also in critical case. We start therefore, with the former constraints.

4.1 The generic case $B \neq B_c$

Let us start with the noncritical case. As in the free case, we pass over to the conjugate complex linear combinations

$$\widetilde{\Lambda}_- = \widetilde{\Lambda}_1 - i\widetilde{\Lambda}_2 \approx 0 \qquad \text{and} \qquad \widetilde{\Lambda}_+ = \widetilde{\Lambda}_1 + i\widetilde{\Lambda}_2 \approx 0. \tag{4.1}$$

The first combination here can be viewed as a first class constraint and the second one as a gauge condition for it. Then, instead of quantizing the system with two second class constraints which generate complicated, field-dependent Poisson-Dirac brackets on reduced phase space, (2.19)–(2.21), we quantize it by the Gupta-Bleuler method. This amounts to using the simple symplectic structure (2.1) in total phase space, and then selecting the physical quantum states by the quantum constraint

$$\tilde{\Lambda}_-|\phi\rangle = 0, \qquad \text{where} \qquad \tilde{\Lambda}_- = P_- - mv_-. \tag{4.2}$$

As a result, we get a correspondence between the classical and quantum descriptions in that, for any two physical states, $\langle\phi_1|\tilde{\Lambda}_\pm|\phi_2\rangle = 0$, where $\tilde{\Lambda}_+ = \tilde{\Lambda}_-^\dagger$.

Any operator $\mathcal{O}$ which leaves the physical subspace (4.2) invariant can be viewed as a quantum observable. Therefore, it has to satisfy a relation of the form

$$[\tilde{\Lambda}_-, \mathcal{O}] = (...)\tilde{\Lambda}_-$$

with the operator Λ_- appearing on the right, cf. (1.3). This happens, in particular, for the operator $Y_i = x_i + m\theta\epsilon_{ij}v_j$, which strongly commutes with the constraint also quantum-mechanically, and is, therefore, a quantum observable.

We have to identify a quantum Hamiltonian, together with two other independent observables [see the classical equation (2.28)]. The quantum counterpart of the 'magnetic Hamiltonian' H_B is chosen to be the Hermitian analog of (2.13), namely

$$H_B = \frac{m}{2} v_+ v_- + u_-^\dagger \tilde{\Lambda}_- + \tilde{\Lambda}_+ u_- . \tag{4.3}$$

The operator-valued coefficient u_- is fixed here by the observability requirement for (4.3) as $u_- = \frac{1}{2}\mathcal{T} v_-$, where the operator $\mathcal{T}$ [further discussed below] is given formally by

$$\mathcal{T} = \left(1 + \frac{1}{2}\frac{\theta}{1-\beta}\tilde{\Lambda}_+\tilde{\Lambda}_-\right)^{-1} \frac{1}{1-\beta} = \frac{1}{1-\beta}\left(1 + \frac{1}{2}\theta\tilde{\Lambda}_+\tilde{\Lambda}_- \frac{1}{1-\beta}\right)^{-1} . \tag{4.4}$$

The quantum analogs of the classical position and momentum operators $\mathcal{X}_\pm$ and $\mathcal{P}_\pm$ are

$$\mathcal{X}_+ = x_+ + i\theta\tilde{\Lambda}_+\mathcal{T}, \qquad \mathcal{X}_- = \mathcal{X}_-^\dagger, \tag{4.5}$$

$$\mathcal{P}_+ = \tilde{\Lambda}_+\mathcal{T} + m v_+, \qquad \mathcal{P}_- = \mathcal{P}_+^\dagger, \tag{4.6}$$

cf. (2.18), (2.16). Minimal and non-minimal coupling, respectively, are obtained adding to H_B the scalar potential with the $\mathcal{X}_i$ resp. Y_i in its argument. Decomposition of the second operator factor in (4.4) into a formal infinite operator series shows, however, that, due to the noncommutativity of $\tilde{\Lambda}_-$ and $\beta(x)$, the theory is in general nonlocal in x_i — except for a homogeneous magnetic field, discussed below.

4.2 Constant magnetic field $B \neq B_c$

Let us assume that the magnetic field is homogeneous $B = const$, $B \neq B_c$. Then the operators $\tilde{\Lambda}_-$ and $(1-\beta)^{-1}$ commute and the action of $\mathcal{T}$ on physical states reduces to multiplication by the constant $(1-\beta)^{-1}$. Thus

$$u_- = \frac{1}{2(1-\beta)} v_-, \tag{4.7}$$

cf. Section 2, and the kinetic Hamiltonian is

$$H_B = \frac{1}{2}\left(\mathcal{P}_+ v_- + v_+ \Lambda_-\right) = \frac{1}{2m}(\mathcal{P}_+\mathcal{P}_- - \Lambda_+\Lambda_-) . \tag{4.8}$$

The observables (4.5) and (4.6) become now local operators

$$\mathcal{X}_+ = x_+ + i\theta\Lambda_+, \qquad \mathcal{X}_- = \mathcal{X}_-^\dagger, \tag{4.9}$$

$$\mathcal{P}_+ = \Lambda_+ + m v_+ = \frac{1}{1-\beta}(P_+ - m\beta v_+), \qquad \mathcal{P}_- = \mathcal{P}_+^\dagger, \tag{4.10}$$

and $\Lambda_\pm = (1-\beta)^{-1}\tilde{\Lambda}_\pm$, cf. the classical relations (2.14), (2.16), and (2.18).

Equation (4.2) means that the physical states are coherent states, namely the eigenstates of the velocity operator v_- with eigenvalue P_-/m. The physical states, defined as solutions of (4.2), are

$$|\phi\rangle_{phys} = \exp\left(\frac{1}{2}\theta m P_- v_+\right)\left(|0\rangle_v|\tilde{\phi}\rangle\right), \tag{4.11}$$

where $|0\rangle_v$, $v_-|0\rangle_v = 0$, is the vacuum state of the Fock space generated by the velocity operators, and $|\tilde{\phi}\rangle$ is a velocity-independent state associated with other degrees of freedom [3].

The action of the observable operators on physical states is reduced to

$$\mathcal{P}_- \to P_-, \qquad \mathcal{P}_+ \to \frac{1}{1-\beta}P_+, \tag{4.12}$$

$$\mathcal{X}_- \to x_-, \qquad \mathcal{X}_+ \to x_+ + i\frac{\theta}{1-\beta}P_+, \qquad Y_- \to x_- + i\theta P_-, \qquad Y_+ \to x_+, \tag{4.13}$$

[3] Various "kets" – distinguished sometimes by lower indices – "live" in different spaces.

in the sense

$$\mathcal{P}_-|\phi\rangle_{phys} = \exp\left(\frac{1}{2}\theta m P_- v_+\right)\left(|0\rangle_v\, P_-|\tilde{\phi}\rangle\right), \tag{4.14}$$

etc., i.e. the operators on the right hand sides act on $|\tilde{\phi}\rangle$. Similarly, the action on physical states of the magnetic Hamiltonian (4.8) and of the quantum analog of the angular momentum (2.22) reduce to

$$H_B \to \frac{1}{2m^*} P_+ P_- \tag{4.15}$$

and

$$J \to \frac{i}{2}(x_+ P_- - x_- P_+) + \frac{1}{2} eB x_+ x_-, \tag{4.16}$$

respectively. On the subspace spanned by the velocity-independent states $|\tilde{\phi}\rangle$, let us define the operators

$$R_+ = P_+ + ieBx_+, \qquad R_- = P_- - ieBx_-, \tag{4.17}$$

which correspond to $\mathcal{P}_+ + ieB\mathcal{X}_+ = ieB\mathcal{X}_+^{gc}$ and $\mathcal{P}_- - ieB\mathcal{X}_- = -ieB\mathcal{X}_-^{gc}$, acting in total space, where $\mathcal{X}_i^{gc}$ is the guiding center coordinate (2.38). On account of the commutation relations

$$[P_+, P_-] = 2eB, \quad [P_-, x_+] = [P_+, x_-] = -2i, \quad [P_+, x_+] = [P_-, x_-] = 0, \tag{4.18}$$

the operators (4.17) commute with P_+, P_- and satisfy the relation

$$[R_+, R_-] = -2eB. \tag{4.19}$$

They reduce the angular momentum operator (4.16) to normal Hermitian form,

$$J \to \frac{1}{2eB}\left(R_+ R_- - P_- P_+\right). \tag{4.20}$$

The commutation relations (4.18) and (4.19) depend on the sign of eB. We have to distinguish therefore two cases. For both signs, we have two independent sets of creation-annihilation oscillator operators $a^{\pm}$ and $b^{\pm}$. The cast is sign-dependent, though :

• For $eB < 0$,

$$a^- = \frac{1}{\sqrt{2|eB|}} P_-, \quad a^+ = \frac{1}{\sqrt{2|eB|}} P_+, \quad b^- = \frac{1}{\sqrt{2|eB|}} R_+, \quad b^+ = \frac{1}{\sqrt{2|eB|}} R_-, \tag{4.21}$$

satisfy $[a^-, a^+] = [b^-, b^+] = 1$, $[a^{\pm}, b^{\pm}] = 0$. In their terms, the angular momentum operator (4.20) takes the canonical quadratic form

$$J \to a^+ a^- - b^+ b^-. \tag{4.22}$$

• For $eB > 0$ we have, instead of (4.21),

$$a^- = \frac{1}{\sqrt{2eB}} P_+, \quad a^+ = \frac{1}{\sqrt{2eB}} P_-, \quad b^- = \frac{1}{\sqrt{2eB}} R_-, \quad b^+ = \frac{1}{\sqrt{2eB}} R_+, \tag{4.23}$$

and the angular momentum operator reads

$$J \to b^+ b^- - a^+ a^-. \tag{4.24}$$

Let us now assume that we have a purely magnetic field. Hence $H = H_B$, and angular momentum is conserved. Consider the physical states (4.11) with

$$|\tilde{\phi}\rangle = |n\rangle_a |l\rangle_b, \qquad \text{where} \qquad a^+ a^- |n\rangle_a = n|n\rangle_a, \quad b^+ b^- |l\rangle_b = l|l\rangle_b,$$

$n, l = 0, 1, \ldots$, i.e. consider the states of the form

$$|n, l) := e^{\frac{1}{2}\theta m P_- v_+} |0\rangle_v |n\rangle_a |l\rangle_b. \tag{4.25}$$

• For $eB < 0$, the (4.25) are eigenstates of operators $H = H_B$ and J with eigenvalues

$$E_N = \frac{|eB|}{m^*}N, \quad N = n = 0, 1, \ldots \qquad \text{and} \qquad j = n - l = N, N-1, \ldots, \tag{4.26}$$

respectively. The energy spectrum is therefore discrete and nonnegative; each Landau level is infinitely degenerate in the angular momentum, which takes integer values bounded from above.

• For $eB > 0$, the roles of the operators P_+ and P_- as creation and annihilation operators are interchanged, and instead of (4.26) we have

$$E_N = \frac{eB}{m^*}(N+1), \quad N = n = 0, 1, \ldots, \qquad \text{and} \qquad j = l - n = -N, -N+1, \ldots. \tag{4.27}$$

Here we should distinguish two further "phases"[4].

•• For $0 < eB < \theta^{-1}$, the energy spectrum is discrete and positive. The Landau levels are infinitely degenerate in j, which takes an infinite number of integer values and is bounded from below.

•• For $eB > \theta^{-1}$, we have the same degeneration of Landau levels, but the effective mass m^* becomes negative. In order to make the theory well defined and to eliminate the negative-energy, unbounded-from-below spectrum, we change the sign of the evolution parameter, $t \to -t$. This is equivalent to changing the sign of the Hamiltonian operator. The energy spectrum is given therefore by (4.27) but with m^* changed into $|m^*|$. Below we shall see, however, that even with such a change of time evolution parameter, the cases $0 < eB < \theta^{-1}$ and $eB > \theta^{-1}$ correspond to the two essentially different phases.

One could attempt to restore the spectrum symmetry around $B = 0$, by changing the quantum ordering. Replacing indeed the quantum Hamiltonian (4.8) by

$$H_B^s = \frac{1}{4}(\mathcal{P}_+ v_- + v_- \mathcal{P}_+) + \frac{1}{2} v_+ \Lambda_- = \frac{1}{4m}(\mathcal{P}_+\mathcal{P}_- + \mathcal{P}_-\mathcal{P}_+) - \frac{1}{2m}\Lambda_+\Lambda_- \tag{4.28}$$

i. e. $H_B^s \to \frac{1}{4m^*}(P_+P_- + P_-P_+)$ on the physical states (4.11) yields, instead of (4.26) and (4.27), the spectrum

$$E_N = \frac{|eB|}{m^*}\left(N + \frac{1}{2}\right), \qquad N = 0, 1, \ldots \tag{4.29}$$

which looks to be symmetric w.r.t. $B = 0$. The asymmetry between $B < 0$ and $B > 0$ is still present, however, since it is hidden in the asymmetric behaviour of the effective mass. To second order in B, the spectrum is indeed $|B|(1 + e\theta B)(N + 1/2)$.

Parity invariance is hence violated by planar noncommutativity, and this is revealed by coupling to a magnetic field.

Let us now investigate the question of normalizability of the states (4.25). Let us define the 'normalized' velocity creation- annihilation oscillator operators

$$c^\pm = \sqrt{\frac{\theta m^2}{2}}\, v_\pm, \qquad [c^-, c^+] = 1, \tag{4.30}$$

and the corresponding Fock states $|k\rangle_v$, $c^+c^-|k\rangle_v = k|k\rangle_v$, $k = 0, 1, \ldots$. Decomposing the exponential factor into Taylor series, we find that for $eB > 0$ the states (4.25) are given by an infinite superposition

$$|n, l) = \sum_{k=0}^{\infty} \beta^{k/2} \sqrt{C_k^{n+k}}\, |k\rangle_v |n+k\rangle_a |l\rangle_b, \qquad C_k^{n+k} = \frac{(n+k)!}{k!\, n!} \tag{4.31}$$

that satisfy the relation

$$(n', l'|n, l) = \delta_{n'n}\delta_{l'l} \sum_{k=0}^{\infty} \beta^k C_k^{n+k}. \tag{4.32}$$

[4] Two phases were discussed also in the context of another noncommutative quantum mechanical model in [16], which is related to the model [11] by a time rescaling, $t \to (m/m^*)t$, supplemented with (nonunitary) change of variables.

• For $0 < eB < \theta^{-1}$, we have $\beta < 1$ and the series in (4.32) converges. The orthonormal Landau states are

$$|n,l\rangle = \mathcal{N}^{-1/2}|n,l), \qquad \mathcal{N} = \mathcal{N}(\beta,n) = \frac{1}{n!}\frac{d^n}{d\beta^n}\left(\frac{1}{1-\beta}\right). \tag{4.33}$$

• For $eB > \theta^{-1}$, $\beta > 1$ and the series in (4.32) diverges. In this case the energy eigenstates (4.31) are scattering-like, unbounded states.

Hence, the two cases $0 < eB < \theta^{-1}$ and $eB > \theta^{-1}$ correspond to two, essentially different phases.

• For $eB < 0$, operator P_- is in fact the annihilation operator a^- [see Eq. (4.21)]. As a result, (4.25) is a superposition of $(n+1)$ states,

$$|n,l) = \sum_{k=0}^{n} |\beta|^{k/2}\sqrt{C_k^n}\,|k\rangle_v|n-k\rangle_a|l\rangle_b. \tag{4.34}$$

The corresponding orthormal states are given by

$$|n,l\rangle = \mathcal{N}^{-1/2}|n,l), \qquad \mathcal{N} = \mathcal{N}(\beta,n) = (1-\beta)^{-n/2}. \tag{4.35}$$

In conclusion, the structure of the states (4.25) is essentially different for $eB < 0$ and $eB > 0$: in the first case it is a normalizable superposition of $(n+1)$ velocity Fock states; in the second case it is an infinite superposition of all velocity-Fock states, which is normalizable (bound state) for $0 < eB < \theta^{-1}$, and is not normalizable (scattering-like state) for $eB > \theta^{-1}$. In this sense the noncomutative Landau problem for $B \neq B_c$ has three, essentially different phases (plus the critical case phase separating the phases $0 < eB < \theta^{-1}$ and $eB > \theta^{-1}$, see below).

The hidden finite resp. infinite dimensional structures of the physical states in the $eB < 0$ and $eB > 0$ phases described here are reminiscent of the finite resp. infinite-dimensional representations of the $sl(2,R)$ algebra, associated with universal Majorana-Dirac-like equations for usual integer/half-integer resp. anyonic fields in $2+1$ dimensions [9], see below.

Note that such a hidden structure would be absent if we quantized the system on the reduced phase space given by the second class constraints. This would eliminate effectively the velocity degrees of freedom ($v_\pm$), and yield the system in Ref. [11], described only by the variables $\mathcal{X}_i$ and $\mathcal{P}_i$, and would have symplectic structure (2.19), (2.20), (2.21). Then, a superposition of the states of the form $|\tilde{\phi}\rangle = |n\rangle_a|l\rangle_b$ would describe the quantum states of the system. In such a quantization scheme it would be impossible to reveal the difference between the energy eigenstates for $eB < \theta^{-1}$ and $eB > \theta^{-1}$ in the sense of their normalizability. This is consistent with an observation [17] which says that different quantization methods can produce physically inequivalent results.

To conclude this section, we generalize the free field equations (1.1)

$$\left\{ \begin{array}{lclr} \left(i\partial_t + eB\dfrac{k}{m^*}\right)\phi_k & + & \sqrt{\dfrac{k+1}{2\theta}}\,\dfrac{P_+}{m^*}\,\phi_{k+1} & = 0, \\[2ex] P_-\,\phi_k & + & \sqrt{\dfrac{2(k+1)}{\theta}}\,\phi_{k+1} & = 0. \end{array} \right. \tag{4.36}$$

Each of these equations is just the component form of the Schrödinger equation $i\partial_t|\phi\rangle = H_B|\phi\rangle$ and constraint equation (4.2), respectively, where H_B is taken in the first, linear-in-$\mathcal{P}$ form in Eq. (4.8), and the field components are defined as $\phi_k(x,t) = (-1)^k({}_v\langle k|\langle x|\phi\rangle)$.

Eliminating the $(k+1)^{th}$ component using the lower equation we find that each component satisfies an independent Schrödinger-Pauli type equation, namely

$$i\partial_t\phi_k = H_k\phi_k, \qquad H_k = H_0 - \frac{eB}{m^*}k, \qquad H_0 = \frac{1}{2m^*}P_+P_-. \tag{4.37}$$

H_0 is, in particular, the restriction of the Landau Hamiltonian H_B to the velocity vacuum sector, cf. Eq. (4.15). For the higher components, the Hamiltonian H_k looks formally like that for a charged particle with planar spin $s = \frac{1}{2}k$ and with gyromagnetic ratio $g = 2$. Our system is

a spinless, however : it is described just by one independent field component ϕ_0, whose internal angular momentum (spin) is zero [1]. It can not be treated as a scalar particle, however, see the last section below. The tower of higher components is generated by ϕ_0 according to

$$\phi_k = (-1)^k \left(\frac{\theta}{2k!}\right)^{k/2} P_-^k \, \phi_0. \tag{4.38}$$

ϕ_0 corresponds to the wave function of the velocity-independent state $|\tilde{\phi}\rangle$ from the physical state (4.11), $\phi_0 = \langle x|\tilde{\phi}\rangle$. In general, it describes an arbitrary superposition of the Landau states $|n,l\rangle$. In the particular case $|\tilde{\phi}\rangle = |n,l\rangle$, the corresponding fields $\phi_k^{n,l}(x,t)$ are the stationary states of the Schrödinger-Pauli equation (4.37). In accordance with Eqs. (4.31) the tower of components $\phi_k^{n,l}$ is infinite for $eB > 0$, while according to Eq. (4.34), for $eB < 0$ it contains a finite number of nonzero components $\phi_k^{n,l}$, $k = 0, 1, ..., n$. In this respect, the structure of our field equations is *formally* similar to that of the infinite-component, relativistic Majorana-Dirac like equations for (2+1)D anyons, and also to the finite- component equations for usual spin $s = \frac{1}{2}n$ fields [9].

5 The critical quantum case

In the critical case the constraints $\widetilde{\Lambda}_i$ in (2.9) are first class. Since the observables Y_i strongly commute with the $\widetilde{\Lambda}_i$, the Hamiltonian can be taken, consistently with Section 3, to be a function of the Y_i, $H_c = \mathcal{H}(Y)$. $\mathcal{H}(Y)$ is given by Eq. (3.13) and (3.16) for minimal or nonminimal coupling, respectively. Here we assume that the observables are realized in terms of the total (initial) phase space variables $Y_i = x_i + m\theta\epsilon_{ij}v_j$, see Eq. (2.28), while the constraints (2.9) should be treated as equations specifying the classical physical subspace. Then, following Dirac, condition (4.2) has to be supplemented at the quantum level with

$$\tilde{\Lambda}_+|\phi\rangle = 0, \qquad \tilde{\Lambda}_+ = P_+ - mv_+. \tag{5.1}$$

Using the solution (4.11) of Eq. (4.2), we find that equation (5.1) is reduced to

$$P_+|\tilde{\phi}\rangle = 0. \tag{5.2}$$

The operators P_+ and P_- satisfy here the commutation relation $[P_+, P_-] = 2\theta^{-1}$, i.e. they act as annihilation and creation operators. Let us choose the gauge

$$A_+ = A_1 + iA_2 = \frac{i}{2e\theta}\, x_+, \qquad A_- = A_1 - iA_2 = -\frac{i}{2e\theta}\, x_-. \tag{5.3}$$

Hence,

$$P_+ = -2i\frac{\partial}{\partial x_-} - \frac{i}{2\theta}x_+, \qquad P_- = -2i\frac{\partial}{\partial x_+} + \frac{i}{2\theta}x_-. \tag{5.4}$$

Then the solution of Eq. (5.2) is given by

$$\tilde{\phi}(x_+, x_-) = \langle x_+, x_-|\tilde{\phi}\rangle = \exp\left(-\frac{1}{4\theta}x_+x_-\right) f(x_+). \tag{5.5}$$

where $f(x_+)$ is an arbitrary holomorphic function. Thus, we recover the wave functions proposed by Laughlin to describe the ground states in the Fractional Quantum Hall Effect [18, 19]. The action of the operators $Y_\pm$ on these functions is reduced to

$$Y_+\tilde{\phi}(x_+, x_-) = e^{-\frac{1}{4\theta}x_+x_-}\, x_+ f(x_+), \qquad Y_-\tilde{\phi}(x_+, x_-) = e^{-\frac{1}{4\theta}x_+x_-}\, 2\theta\frac{d}{dx_+}f(x_+), \tag{5.6}$$

cf. (4.13).

Due to the noncommutativity of the operators Y_- and Y_+, the construction of the quantum Hamiltonian $\mathcal{H}(Y)$ requires to choose an ordering.

Let us clarify now how such a quantum picture is related to the generic Landau problem with a noncritical magnetic field. First we note that, just like classically, the critical case corresponds to the boundary between two different phases, namely $0 < eB < \theta^{-1}$ and $eB > \theta^{-1}$, respectively. In accordance with Eqs. (4.12), (4.13), the Y_i are (unlike $\mathcal{P}_+$ and $\mathcal{X}_+$) well defined at $eB = \theta^{-1}$:

$$Y_- \to i\sqrt{2\theta}\, b^-, \qquad Y_+ \to -i\sqrt{2\theta}\,(b^+ - a^-).$$

Since at $eB = \theta^{-1}$ the second constraint equation is reduced to Eq. (5.2), we find that the physical states now belong to the lowest Landau level (LLL),

$$|0, l\rangle = e^{\frac{1}{2}\theta m P_- v_+}|0\rangle_v |0\rangle_a |l\rangle_b. \tag{5.7}$$

By Eq. (4.32), such states are not normalizable. We can 'renormalize' the theory by first projecting, in the phase $0 < eB < \theta^{-1}$, to the lowest Landau level, and then taking the limit $\beta \to 1$. The projection of an observable $\mathcal{O}$ to the lowest Landau level is

$$(\mathcal{O})_0 = \Pi_0 \mathcal{O} \Pi_0 \qquad \text{where} \qquad \Pi_0 = \sum_{l=0}^{\infty} |0, l\rangle\langle 0, l| \tag{5.8}$$

[for $N = n = 0$, $j = l$, see Eq. (4.27)]. Taking into account the definitions (4.17) and (4.23), we find that

$$(\mathcal{P}_\pm)_0 = 0, \quad (\mathcal{X}_-)_0 = (Y_-)_0 = i\sqrt{\frac{2}{eB}}\, b^- \,\Pi_0, \quad (\mathcal{X}_+)_0 = (Y_+)_0 = -i\sqrt{\frac{2}{eB}}\, b^+ \,\Pi_0. \tag{5.9}$$

Therefore, when restricted to the lowest Landau level, the coordinates $\mathcal{X}_i$ act identically as the Y_i do. Letting $\beta \to 1$ yields that, in the critical case, the action of these operators is reduced to

$$(\mathcal{X}_-)_0 = (Y_-)_0 \longrightarrow i\sqrt{2\theta}\, b^- \qquad (\mathcal{X}_+)_0 = (Y_+)_0 \longrightarrow -i\sqrt{2\theta}\, b^+, \tag{5.10}$$

cf. Eq. (5.6), where, on the right hand sides we assume that the operators act on LLL states. These operators are the only independent nontrivial operators of the system put into a critical magnetic field, and, in accordance with Eqns. (4.17) and (4.23), are the guiding center coordinate (2.38) reduced to the LLL states. Note that the relations (5.10) correspond to the classical brackets (3.12) taken for $V = 0$ and with $x_i \approx Y_i$, valid generically for nonminimal coupling with $B = B_c$ (see Section 3).

The normalized states (4.33) corresponding to the lowest Landau level for $0 < eB < \theta^{-1}$ are

$$|0, l\rangle = \left((1 - \beta)^{1/2} \sum_{k=0}^{\infty} \beta^{k/2} |k\rangle_v |k\rangle_a \right) |l\rangle_b.$$

Taking the limit $\beta \to 1$ and factorizing the fixed state vector in the bracket yields the reduced quantum space of those LLL states $|l\rangle_b$.

The Laughlin wave functions (5.5) are the states $|l\rangle_b$ written in holomorphic representation, in which the operators (5.10) are reduced to (5.6).

6 Discussion and concluding remarks

The main result of this paper is a consistent scheme to couple the first-order, Majorana-Dirac type equations that describe a non-relativistic anyon to an electromagnetic field. (An alternative framework is presented in [20].) Let us stress that the analogous results in the relativistic setting are still missing[5].

Our framework here is analogous to the usual electromagnetic coupling of a Dirac particle in terms of the initial coordinates x_i, subject to Zitterbewegung. That in [11] corresponds in turn to

[5]For relativistic models of anyons see [5, 7, 8, 9, 21].

the introduction of electromagnetic interaction using the Zitterbewegung-free Foldy-Wouthuysen coordinates.

The Dirac equation for a spin-1/2 particle can be treated as a constraint which removes the timelike spin degrees of freedom (which otherwise would produce negative norm states), and generates the mass shell, Klein-Gordon equation. The structure behind both of these properties is the local supersupersymmetry of a Dirac particle [22]. Modification of a free Dirac equation according to the minimal coupling prescription produces, via local supersymmetry, the correct form of the mass shell equation.

For our system, the constraint makes, in the free case, the spectrum bounded from below by freezing its spin degrees of freedom. So, it plays the role similar to that of the (first-order) Dirac equation, while the Schrödinger equation is analogous to the (quadratic) mass-shell condition. We have, however, no local supersymmetry here. To switch on interaction we use instead weak commutativity as guiding principle. This fixes the form of the Hamiltonian to be consistent with the constraint, modified according to the minimal coupling prescription.

When the magnetic field is non-constant (inhomogeneous), our quantum theory becomes nonlocal. This property is actually inherited from relativistic anyons from which our theory can be derived [1]. It is indeed reminiscent of the nonlocality of anyon fields within the framework of the Chern-Simons U(1) gauge field construction, with associated half-infinite nonobservable 'string', [23, 24, 25].

Viewing the critical case also as a "phase", we can say that, in the non-commutative Landau problem, our system exhibits *four* different phases. These "phases" may be viewed as analogous to various energy sectors of the Kepler problem,

- $E < 0 \qquad (eB < \theta^{-1})$,
- $E = 0 \qquad (eB = \theta^{-1})$,
- $E > 0 \qquad (eB > \theta^{-1})$

with its changing $SO(4)$, $E(3)$ and $SO(3,1)$ symmetry.

The phases on each side of $B = 0$ exhibit a spectral asymmetry with respect to the sign of magnetic field, and an essentially different structure of the states: for $eB < 0$, the angular momentum takes an infinite number of values, $j = N, N-1, \ldots, -\infty$. The field associated with a Landau state of level N has $(N+1)$-components. On the other hand, for $0 < eB < \theta^{-1}$, the field is infinite-component, with the angular momentum taking $j = -N, -N+1, \ldots, +\infty$, for a level-$N$ Landau state. In both phases the corresponding quantum mechanical states are normalizable and represent bound states.

In the phase $eB > \theta^{-1}$, for a Landau state of level N, the angular momentum can take an infinite half-bounded set of integer values, with, for any fixed value of j, an associated infinite-component field. The corresponding quantum states are non-normalizable scattering states.

In the critical case $eB\theta = 1$, our framework generalized to the non-commutative context the Peierl's substitution [14] and Laughlin's framework for the Fractional Quantum Hall Effect [19].

In the free case ($\mathcal{P}_i = p_i$), our system describes a spinless particle. It cannot be interpreted, however, as an "ordinary" scalar particle. In the rest frame system ($\vec{p} = 0$) it is described effectively by the velocity vacuum state (see Eqn. (4.11)) and by associated one field component only (constant in this frame). In a boosted frame, however, an infinite tower of velocity Fock states contribute, coherently, to the physical state; an infinite tower of field components (4.38) is "brought to life". Thus, the internal spin degrees of freedom associated with velocity operators (see Eqn. (2.4)) are here frozen, rather then removed as in the model [11]. The corresponding velocity Fock states $|k\rangle_v$ with $k > 0$ can be compared with a Dirac sea of the negative energy states.

When we switch on a magnetic field, the 'frozen sea' 'melts'. For $eB < 0$, in the state corresponding to the N^{th} Landau level, the first N exited velocity Fock states contribute to the state. In contrast, for $eB > 0$, the entire, infinite set of velocity Fock states is "brought to life", just like in the free case for $\vec{p} \neq 0$. Taking also into account the normalizability properties of the physical states, the phases with $eB < 0$, $0 < eB < \theta^{-1}$, $eB > \theta^{-1}$ and $eB = \theta = 1$ are somewhat reminiscent resp. to a partially 'melted', a 'liquid', a 'gaseous' and 'boiling' phases.

The observed spectral asymmetry of the system is reminiscent of the phase transition between

exact (for $eB < 0$) and spontaneously broken (for $eB > 0$) supersymmetry, see [26]. An analogous asymmetry has been observed earlier in noncommutative Chern-Simons theories, namely between self-dual and anti-self-dual solution, see [27].

In the noncritical case, we only analysed in detail the purely magnetic Landau problem. Adding a potential term $V(\mathcal{X})$ (or $\check{V}(Y)$), the analog of first equation in (4.36) would involve other field components, due to the presence of velocity operators in operators $\mathcal{X}_i$ and Y_i, see Eqns. (4.9), (2.28). The second equation in (4.36), which controls the number of nontrivial field components would not change, however. These latter behave, therefore exactly in the same way as in the pure Landau problem.

Acknowledgements. MP is indebted to the *Laboratoire de Mathématiques et de Physique Théorique* of Tours University for hospitality extended to him. The partial support by the FONDECYT, Chile (Grants 1010073 and 7010073) is acknowledged.

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